Exam 1 review. CSI 30. Spring 2024. Prof. Pineiro

1. Write the propositions using simpler propositions and logical connectives and determine, if possible, true or false.

   (a) If Philadelphia is the capital of the US then $7 < 3$

   (b) If you do not like to come to school then the number 4 is even.

   (c) The water is solid if and only if $4 < 2$.

   (d) 5 is a prime number or 6 is divisible by 4.

   (a) p: Philadelphia is the capital of the US and q: $7 < 3$. Then: $p \to q$ is True since p is false.

   (b) p: you do not like to come to school and q: 4 is even. Then: $p \to q$ is True since q is true.

   (c) p: water is solid and q: $4 < 2$. Then: $p \to q$ is True since both p and q are false.

   (d) p: 5 is a prime number and q: 6 is divisible by 4. Then: $p \vee q$ is True since p is True. (On the other hand $p \wedge q$ will be False since q is False.)

2. Let p be the proposition " $2 \leq 5$", q the proposition "8 is an even number," and r "11 is a prime number." Express the following as a statement in English and determine whether the statement is true or false:

   (a) $\neg p \wedge q$: $2 > 5$ and 8 is even. (F)

   (b) $p \to q$: If $2 \leq 5$ then 8 is an even number. (T)

   (c) $(p \wedge q) \to r$: If $2 \leq 5$ and 8 is even then 11 is prime. (T)

   (d) $p \to (q \vee (\neg r))$: If $2 \leq 5$ then 8 is even or 11 is not prime. (T)

   (e) $(\neg q) \to (\neg p))$: If 8 is not an even number then $2 > 5$. (T)

3. Translate into a logical expression: "You cannot fly on a plane if you are under 10 unless you come with your parents."
   Let p: you can fly on a plane, q: you are under 10 and r: you come with your parents. Then we have:

   $$(q \wedge \neg r) \to \neg p \qquad \text{or alternative} \qquad \neg r \to (q \to \neg p).$$

   You can use De Morgan's rules to see that both propositions above are equivalent to:

   $$r \vee \neg q \vee \neg p.$$

4. Show that the compound proposition

   $$(\neg q \wedge (p \to q)) \to \neg p$$

is a tautology using a truth table.

p	q	$\neg p$	$\neg q$	$p \to q$	$(\neg q \wedge (p \to q))$	$(\neg q \wedge (p \to q)) \to \neg p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

5. Show that the compound proposition

$$(\neg q \wedge (p \to q)) \to \neg p$$

is a tautology **without using** a truth table.
You can simply invoke Modus Tolens. An alternative solution would be:

$$\begin{aligned}
(\neg q \wedge (p \to q)) \to \neg p &\equiv \neg q \wedge (\neg p \vee q) \to \neg p && \text{(Definition)} \\
&\equiv (\neg q \wedge \neg p) \vee (\neg q \wedge q) \to \neg p && \text{(Distributivity)} \\
&\equiv (\neg q \wedge \neg p) \vee F \to \neg p && \text{(Negation)} \\
&\equiv (\neg q \wedge \neg p) \to \neg p && \text{(Identity)} \\
&\equiv \neg(\neg q \wedge \neg p) \vee \neg p && \text{(Definition)} \\
&\equiv (\neg\neg q \vee \neg\neg p) \vee \neg p && \text{(De Morgan's)} \\
&\equiv (q \vee p) \vee \neg p && \text{(Double negations)} \\
&\equiv q \vee (p \vee \neg p) && \text{(Associativity)} \\
&\equiv q \vee T && \text{(Negation)} \\
&\equiv T && \text{(Domination)}
\end{aligned}$$

6. Show that the compound proposition

$$(p \wedge (p \to q)) \to q$$

is a tautology **without using** a truth table.

7. Show that the argument with premises:

$$(p \wedge t) \to (r \vee s), \quad q \to (u \wedge t), \quad u \to p, \quad \neg s, \quad q$$

and conclusion $q \to r$ is a valid argument.

Step	Rule	Reason
1	q	Premise
2	$q \to (u \wedge t)$	Premise
3	$u \wedge t$	Modus Ponens
4	u	Simplification
5	t	Simplification
6	$u \to p$	Premise
7	p	Modus Ponens
8	$p \wedge t$	Conjunction
8	$(p \wedge t) \to (r \vee s)$	Premise
9	$r \vee s$	Modus Ponens
10	$\neg s$	Premise
11	r	Disjunctive Syllogism

8. Show that $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$ are logically equivalent. We do the sequence of logical equivalences:

$$\begin{aligned}(p \to q) \vee (p \to r) &\equiv (\neg p \vee q) \vee (\neg p \vee r) && \text{(Definition)}\\ &\equiv (\neg p \vee \neg p) \vee (q \vee r) && \text{(Associativity)}\\ &\equiv (\neg p) \vee (q \vee r) && \text{(Idempotent)}\\ &\equiv p \to (q \vee r) && \text{(Definition)}\end{aligned}$$

9. Use De Morgan's Laws to find **the negation** of each of the following statements.

   (a) Pau is doing this semester Calculus I or Calculus II.
   Ans: Pau is not doing this semester neither Calculus I nor Calculus II. This semester, he is not in Calc I and he is not in Calc II.
   (b) Pau is doing this semester Calculus I and Calculus II.
   Ans: This semester, Pau is either not doing Calc I or not doing Calc II. (Pau is not doing both: Calc I and II this semester).
   (c) Every person has a subject that she/he enjoys.
   Ans: There are people that do not enjoy any subject. (There exist at least one person for whom no subject is enjoyable).

10. Express each of the following statements using predicates, quantifiers, logical connectives, and mathematical operators. Domain: all real numbers.
    (a) 'There exists a real number such that if any real number is multiplied by it, we get 0.

$$\exists\, x\, \forall\, y \;\; xy = 0.$$

    (b) 'For every real number, there is a number above it'.

$$\forall\, x\, \exists\, y \;\; y > x.$$

    (c) 'For every real number, there is a number below it'.

$$\forall\, x\, \exists\, y \;\; y < x.$$

11. Let $D(x, y)$ be the statement "The number x divides the number y". Express, using quantifiers and logical connectives, the expression

$$P(x): \quad \text{“The number } x \text{ is a prime number”.}$$

Ans: $P(x) : \neg(x = 1) \wedge (\forall y(D(y, x) \rightarrow (y = x \vee y = 1)))$.

12. Rewrite the statement

$$\neg\exists y\,(\forall x\exists z T(x, y, z) \vee \exists z U(x, y, z))$$

so that negations appear only within predicates:

$$\begin{aligned}\neg\exists y\,(\forall x\exists z T(x, y, z) \vee \exists z U(x, y, z)) &\equiv \forall y \neg\,(\forall x\exists z T(x, y, z) \vee \exists z U(x, y, z)) \\ &\equiv \forall y\,(\neg(\forall x\exists z T(x, y, z)) \wedge \neg(\exists z U(x, y, z))) \\ &\equiv \forall y\,(\exists x\forall z \neg T(x, y, z) \wedge \forall z \neg U(x, y, z))\end{aligned}$$

13. Let $I(x)$ be the statement "x has an Internet connection" and $C(x, y)$ be the statement "x and y have chatted over the Internet", where the domain for the variables x and y consists of all students in your class. Use quantifiers to express each of these statements.
(a) There are two students in your class who have not chatted with each other over the Internet. $\exists x, y\, \neg C(x, y)$.
(b) There are two students in your class who have internet connection and not chatted with each other over the Internet. $\exists\, x, y\, I(x) \wedge I(y) \wedge \neg C(x, y)$
(c) There is a student in your class who has chatted with everyone in your class over the Internet. $\exists\, x\, \forall\, y\, C(x, y)$
(d) If a student in the class has chatted with someone else then this student has an internet connection. $\forall\, x\, (\exists\, y\, C(x, y) \rightarrow I(x))$.

14. Write the converse, inverse and the contrapositive of the statement "If you are a Computer Science major, then you know Discrete Mathematics."
converse: "If you know Discrete Mathematics, then you are a Computer Science major."
inverse: "If you are not a Computer Science major, then you do not know Discrete Mathematics"
contrapositive: "If you do not know Discrete Mathematics, then you are not a Computer Science major."
We observe that a proposition $p \rightarrow q$ and its contrapositive $\neg q \rightarrow \neg p$ are logically equivalent. In the same way the inverse $\neg p \rightarrow \neg q$ and the converse $q \rightarrow p$ are also logically equivalent among themselves but not to the original $p \rightarrow q$.

15. Here are three premises:
   1. Every bird has two feet.
   2. Every insect has six feet.

3. Polly has two feet.

If we conclude "Polly is a bird," have we made a valid argument? If not, why not?

If p: x is a bird, q: x has two feet, r: x is an insect, s: x has six feet. We have the implications:

$$p \to q, \qquad r \to s$$

If we add the premise q for Polly. Can we get p out of those? No, the implication $p \to q$ will be true independently of whether p is true or not. We cannot made a valid argument that gives p based on those premises.